Exact triangles from Dehn twist (ref. Seidel's book §17) 10/20/11

(I) Psuedo-hol. sections (compact base)

$\boxed{\text{Def}}$ $\pi: E^{2n+2} \to S$ exact Lag. fibration, a

Lag. bdry condition is a submfd F^{n+1} of E,

$F \subseteq \partial^v E$, s.t. $\exists\, \alpha_F \in \Omega^1(\partial S)$, $h_F \in C^\infty(F, \mathbb{R})$, s.t.

$$\theta_E|_F = (\pi|_F)^* (\alpha_F + \theta_S|_{\partial S}) + dh_F$$

observe: → F is Lagrangian

→ $\forall z \in \partial S$, F_z exact Lag., carried into each other by // transport along ∂S.

$\boxed{\text{Def}}$ (Fix a Lag. bdry condi. F).

A rel. perturbation datum is (K, J)

* $K \in \Omega^1(E)$, $K(TE^v) = 0$, $K|_F = 0 \in \Omega^1(F)$

  $K = 0$ near $\partial^h E \cup \mathrm{Crit}(\pi)$

* J, ω_E compatible a.c.s. s.t. π is psuedo-hol.

  $J = I_E$ near $\partial^h E \cup \mathrm{Crit}(\pi)$

An inhomogeneous psuedo-hol section w/ bdry in F

is $u: S \to E$ w/

① $\pi(u(z)) = z$

② $Du(z) + J(u) \circ D(u(z)) \circ I_S = Y(u) + J(u) \circ Y(u) \circ I_S$

③ $u(\partial S) \subseteq F$

Y determined by K: a section of $\mathrm{Hom}(\pi^* TS, TE^v)$

$$(X_K)_x = Y(\xi)_x$$

Ham. vect. field on E_z of $K|_{E_z}$

$\pi(x) = z \in S$

$\xi \in (TS)_z$

$\mathcal{M}_{E/S}$ = moduli space $\{u: S \to E \mid ①, ②, ③\}$

— Take generic (K, J), so it's a closed mfd

$\Phi_{E/S} = \# \mathcal{M}^0_{E/S} \in \mathbb{Z}/2$

$\boxed{\text{Ex}}$ (Local model)

$E = \{x \in \mathbb{C}^{n+1} \mid Q(x) \leq r,\ k(x) \leq s\}$

$\downarrow$ $Q: (x_1, \ldots, x_{n+1}) \mapsto (\sum_1 x_i^2) = z$

$D^2(r)$ $\qquad \theta_E = \frac{i}{4} \sum_j x_j d\bar{x}_j - \bar{x}_j dx_j$

* Lag. bdry condition:

  $F_z = \sqrt{z}\, S^n$, $|z| = r$, vanishing cycles, preserved by //. transport

  $F = \bigcup_{|z|=r} F_z$.

* Take $(K, J) = (0, I_E)$

* Psuedo-hol. sections w/ bdry in F:

  ②,③ $\Rightarrow u_a: S \to E$, $u_a(z) = \frac{1}{\sqrt{r}} a z + \sqrt{r}\, \bar{a}$, $a \in \mathbb{C}^{n+1}$

  ① $\Rightarrow a_1^2 + \cdots + a_{n+1}^2 = 0$, $\|a\|^2 = \frac{1}{2}$

  ($\Leftrightarrow \|\mathrm{Re}\, a\|^2 = \|\mathrm{im}\, a\|^2 = \frac{1}{4}$, $\langle \mathrm{re}\, a, \mathrm{im}\, a \rangle = 0$)

* $\dim \mathcal{M}_{E/D^2(r)} = \mathrm{ind}\, D_{\bar\partial} = 2n-1$

  no isolated pts

  $\Phi_{E/D^2(r)} = 0$

$\boxed{\text{Lemma}}$ $\Phi_{E/S} = 0$, S a disc with a single critical pt, $F_z = V$ for $z \in \partial S$.

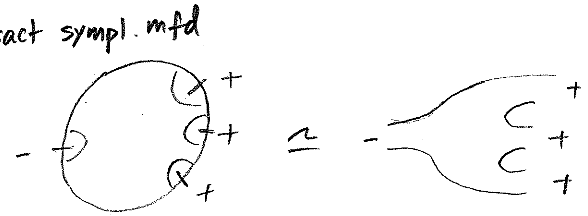
Reason: Use cobordism argument to compare it to the local model.

Another reason: if asume $2c_1(M, V) = 0 \in H^2(M, V)$, can compute

$$\dim \mathcal{M}_{E/S} = \mathrm{ind}(D_{E/S, u}) = 2n-1 \text{ directly}$$

(II) Psuedo-hol. sections (w/ strip-like ends)

Def Lef. fibration w/ strip-like ends: fiber M, exact sympl. mfd



$$\begin{array}{ccc} Z^{\pm}\times M & & E \\ \downarrow & & \downarrow \\ \varepsilon_\xi : Z^{\pm} & \longrightarrow & U_\xi \subseteq S \end{array}$$

π trivial on strip-like ends

$Z^{\pm} = \mathbb{R}^{\pm}\times[0,1]$

S has marked pts $\{\xi\}_{\xi\in\Sigma_1^{\pm}}$

U_ξ = (nbhd of ξ) $\setminus \{\xi\}$

Def Lagr. bdry condition

Take $F \subseteq E \setminus \pi^{-1}(\{\text{mark pts}\})$

$\exists\ (L_{\xi,0}, L_{\xi,1})$ s.t. $F_{\varepsilon_\xi(s,k)} = L_{\xi,k}\quad \forall s \quad (k=0,1)$

$\Rightarrow \alpha_F = 0$ on strip-like ends

Def (Fix F. Fix Floer datum for each pair $(L_{\xi,0}, L_{\xi,1})$)

A relative perturb. datum (K,J) as before, w/

additional restriction over strip-like ends: $K(s,t,x) = H_\xi(t,x)\,dt$, $J(s,t,x) = i\times J_\xi(t,x)$
(they are translation invariant)

H and J are Floer data for $(L_{\xi,0}, L_{\xi,1})$ as psuedohol curve eqn and Floer eqn match for $u: Z_\xi^{\pm}\to E$ on strip-like ends.

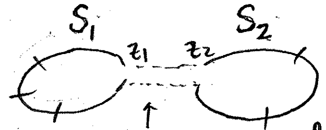
Set $\mathcal{M}_{E/S}(\{y_\xi\})$ = moduli space $\{u,\ \lim_{s\to\pm\infty} u(\varepsilon_\xi(s,\cdot)) = y_\xi,\ y_\xi\in\mathcal{C}(L_{\xi,0}, L_{\xi,1})\}$

$$C\Phi_{E/S} : \bigotimes_{\xi^+\in\Sigma_1^{+}} CF^{pr}(L_{\xi^+,0}, L_{\xi^+,1}) \longrightarrow \bigotimes_{\xi^-\in\Sigma_1^{-}} CF^{pr}(L_{\xi^-,0}, L_{\xi^-,1})$$

$$\Big(\bigotimes_{\xi^+\in\Sigma_1^{+}} y_{\xi^+}\Big) \longmapsto \sum_{\{y_\xi^-\}} \underset{\mathbb{Z}/2}{\underset{\cap}{\#\,\mathcal{M}^0_{E/S}(\{y_\xi^-, y_\xi^+\})}}\ \Big(\bigotimes_{\xi^-\in\Sigma^-} y_{\xi^-}\Big)$$

Gluing:

$$\begin{array}{ccc} E_1 & & E_2 \\ \downarrow & & \downarrow \\ S_1 & & S_2 \end{array}$$



use strip-like end to glue

need: $(E_1)_{z_1} = M = (E_2)_{z_2}$
$(F_1)_{z_1} = \underset{\cup\!\shortmid}{L} = (F_2)_{z_2}$

Theorem $\mathcal{M}_{E/S}(\{y_\xi\})^0 = \coprod_{p+q=n} \mathcal{M}^p_{E_1/S_1}(\{y_\xi\}) \times_L \mathcal{M}^q_{E_2/S_2}$

use ev_{z_1}, ev_{z_2} to take this fiber product

Lemma $\Big(\underbrace{\Phi_{E_2/S_2, z_2}}_{[ev_{z_2}: \mathcal{M}_{E_2/S_2}\to L]\in H^*(L)} = 0\Big) \Rightarrow \big(\Phi_{E/S} = 0\big)$

there's a choice of (K,J) s.t. $C\Phi_{E/S} = 0$

RMK on application (of Lemma)

* Take S_2 compact disk, one crit. value inside, $L=V$
  Assume $2c_1(M,V) = 0$
  $n>1$: $\dim \mathcal{M}_{E_2/S_2} = 2n-1 > n \Rightarrow \mathcal{M}_{E/S}(\{y_\xi\})^0 = \phi$
  $n=1$: $q=1$, $p=0$
  $u_1\in\mathcal{M}_{E_1/S_1}(\{y_\xi\})^0$ is matched by an even # of $u_2\in\mathcal{M}'_{E_2/S_2}$ by eval. maps, $ev_{z_1}(u_1) = ev_{z_2}(u_2)$
  So fiber product has even # of points.

(III) Construction of Morphism

$2c_1(M,V)=0$ (for applying vanishing lemmas)

$L_0=V$, L_1, L_2 exact Lagr., $L_2 \overset{\text{isotopy}}{\simeq} \tau_V(L_1)$

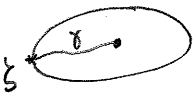
Fix Floer data for (L_0,L_1), (L_0,L_2), (L_1,L_2)

& perturbation data for defining $\mu^2: CF^{pr}(L_1,L_2)\otimes CF^{pr}(L_0,L_1)\to CF^{pr}(L_0,L_2)$

Fix Lef. fibration $\pi: E \to S$ w/ one critical pt and vanishing cycle V.

(can construct Lef. fibr. s.t. vanishing cycle of any vanishing path is V; monodromy for any loop wind around crit val. once is τ_V)

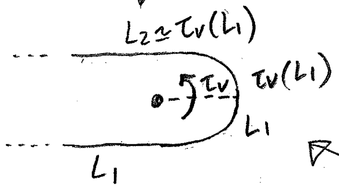
$\partial S \ni \zeta$



Construct a Floer cocycle:

$c \in CF^{pr}(L_1,L_2)$

$\mu^1(c)=0$ b/c ∂ of moduli sp of sections is broken strip $\mu^1(c)$



$E^c = E\setminus \pi^{-1}\{\zeta\}$
$\downarrow$
$S^c = S\setminus\{\zeta\}$
has Lag. bdry condition

$c = C\Phi_{E^c/S^c}$

Construct a chain homotopy b/n $\mu^2(c,\cdot)$ and zero map

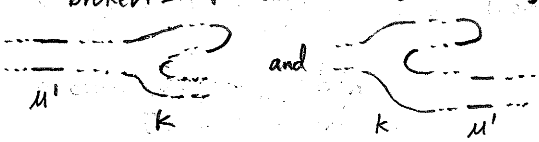
$K: CF^{pr}(L_0,L_1)\longrightarrow CF^{pr}(L_0,L_1)$

$\mu^1(K(\cdot))+K(\mu^1(\cdot))+\mu^2(c,\cdot)=0$

* Construct a family $r\in[0,1]$

$E^{k,r} \to S^{k,r}$ of Lefschetz fibr. w/ bdry cond. $F^{k,r}$, a family of perturb. data.

* other isolated pts in the parametrized moduli space of sections are config. w/ broken strips at some of $\{r\in(0,1)\}$

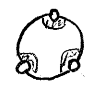


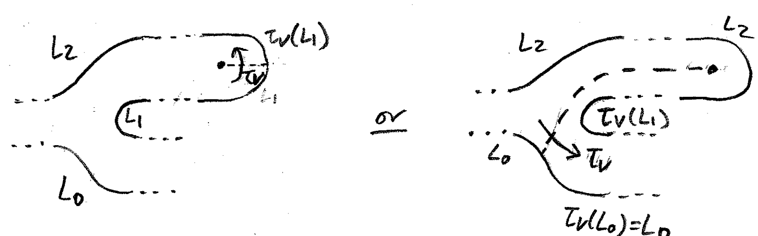
get $K: CF^{pr}(L_0,L_1)\to CF^{pr}(L_0,L_2)$ by counting these.

* Choices are made in this construction, but if (c,K), $(\tilde c,\tilde K)$ are two different choices, then

$c=\tilde c$ up to homotopy given by μ^1

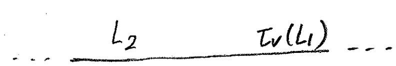
$K=\tilde K$ " " " " " μ^1,μ^2

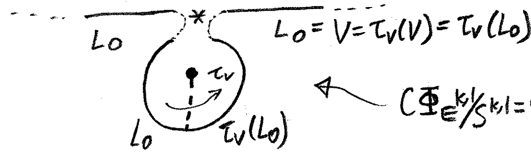
$\underline{r=0}$ trivial fibration over , Lag. bdry cond. (L_0,L_1,L_2) $F^{1,0}, F^{2,0}, F^{3,0}$



$$C\Phi_{E^{k,0}/S^{k,0}} = \mu^2(c,\cdot)$$

$\underline{r=1}$ Trivial fibration over Z,





$C\Phi_{E^{k,1}/S^{k,1}}=0$ due to this compact part